SADLER UNIT 3 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 7: Vector calculus

Exercise 7A

Question 1

Given $\mathbf{r} = 2t^3\mathbf{i} + (3t+1)\mathbf{j}$

a The initial position vector occurs at $t = 0$.

$\mathbf{r} = 2(0)^3\mathbf{i} + (3\times 0+1)\mathbf{j}$

$\mathbf{r} = \mathbf{j}$ m

b $\mathbf{v} = 6t^2\mathbf{i} + 3\mathbf{j}$

When $t=3$, velocity of the particle is $6\times 3^2\mathbf{i} + 3\mathbf{j}$.

The particle has a velocity of $(54\mathbf{i} + 3\mathbf{j})$ m/s when $t=3$.

c $|\mathbf{v}| = \sqrt{54^2+3^2} = 15\sqrt{13}$

The speed of the particle when $t=3$ is $15\sqrt{3}$ m/s.

d $\mathbf{a} = 12t\mathbf{i}$

When $t=3$ the acceleration of the particle is $36\mathbf{i}$ m/s^2.

Question 2

$\mathbf{a} = 6t\mathbf{i}$ m/s^2

a $\mathbf{v} = \int 6t\mathbf{i}\,dt = 3t^2\mathbf{i} + \mathbf{c}$

When $t = 0$, $\mathbf{v} = -4\mathbf{i} + 6\mathbf{j}$, so $\mathbf{c} = -4\mathbf{i} + 6\mathbf{j}$

$\mathbf{v} = \left[(3t^2 - 4)\mathbf{i} + 6\mathbf{j}\right]$ m/s

When $t = 2$, $\mathbf{v} = \left[(3 \times 4 - 4)\mathbf{i} + 6\mathbf{j}\right]$ m/s

$\mathbf{v} = (8\mathbf{i} + 6\mathbf{j})$ m/s

$|\mathbf{v}| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$ m/s

b $\mathbf{r} = \int \left[(3t^2 - 4)\mathbf{i} + 6\mathbf{j}\right] dt = (t^3 - 4t)\mathbf{i} + 6t\mathbf{j} + c$

When $t = 0$, $\mathbf{r} = (2\mathbf{i} + \mathbf{j})$ m

$c = (2\mathbf{i} + \mathbf{j})$

$\mathbf{r} = (t^3 - 4t)\mathbf{i} + 6t\mathbf{j} + 2\mathbf{i} + \mathbf{j} = (t^3 - 4t + 2)\mathbf{i} + (6t + 1)\mathbf{j}$

When $t = 2$, $\mathbf{r} = 2\mathbf{i} + 13\mathbf{j}$

The distance from $2\mathbf{i} + \mathbf{j}$ to $2\mathbf{i} + 13\mathbf{j}$ is $\sqrt{(2-2)^2 + (13-1)^2} = 12$ m.

Question 3

$\mathbf{r} = 2t\mathbf{i} + (t - 1)\mathbf{j}$

a $\dfrac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j}$

$\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{2^2 + 1^2} = \sqrt{5}$

b $|\mathbf{r}| = \sqrt{(2t)^2 + (t-1)^2} = \sqrt{4t^2 + t^2 - 2t + 1} = (5t^2 - 2t + 1)^{\frac{1}{2}}$

$\dfrac{d}{dt}|\mathbf{r}| = \dfrac{1}{2}(5t^2 - 2t + 1)^{-\frac{1}{2}}(10t - 2) = \dfrac{10t - 2}{2\sqrt{5t^2 - 2t + 1}} = \dfrac{5t - 1}{\sqrt{5t^2 - 2t + 1}}$

Question 4

a $\mathbf{v}(1) = \frac{-1}{(1+1)^2}\mathbf{i} + 2\mathbf{j} = -0.25\mathbf{i} + 2\mathbf{j}$

b $\mathbf{a}(t) = \frac{d}{dt}\left(\frac{-1}{(t+1)^2}\mathbf{i} + 2\mathbf{j}\right) = \frac{2}{(t+1)^3}\mathbf{i}$

$\mathbf{a}(1) = \frac{2}{(1+1)^3}\mathbf{i} = \frac{1}{4}\mathbf{i}$

c $\mathbf{r}(t) = \int\left[\frac{-1}{(t+1)^2}\mathbf{i} + 2\mathbf{j}\right]dt = \int\left[-(t+1)^{-2}\mathbf{i} + 2\mathbf{j}\right]dt = (t+1)^{-1}\mathbf{i} + 2t\mathbf{j} + c$

$\mathbf{r}(0) = 3\mathbf{i} + 3\mathbf{j}$

$(1)^{-1}\mathbf{i} + 2(0)\mathbf{j} + c = 3\mathbf{i} + 3\mathbf{j}$

$c = 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{r}(t) = \frac{1}{t+1}\mathbf{i} + 2t\mathbf{j} + 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{r}(1) = \frac{1}{1+1}\mathbf{i} + 2\mathbf{j} + 2\mathbf{i} + 3\mathbf{j} = 2.5\mathbf{i} + 5\mathbf{j}$

Question 5

a $\mathbf{r} = (t^2 - 5t + 1)\mathbf{i} + (1 - 14t + t^2)\mathbf{j}$

$\mathbf{v} = (2t - 5)\mathbf{i} + (2t - 14)\mathbf{j}$

To be travelling parallel to the x-axis the $\mathbf{j}$ component of the velocity vector must be zero.

This occurs when $2t - 14 = 0$.

The particle is travelling parallel to the x-axis when $t = 7$ seconds.

b $\mathbf{r} = (t^2 - 5t + 1)\mathbf{i} + (1 - 14t + t^2)\mathbf{j}$

$\mathbf{v} = (2t - 5)\mathbf{i} + (2t - 14)\mathbf{j}$

To be travelling parallel to the y-axis the $\mathbf{i}$ component of the velocity vector must be zero.

This occurs when $2t - 5 = 0$.

The particle is travelling parallel to the y-axis when $t = 2.5$ seconds.

Question 6

a $\mathbf{v}(10) = 2\mathbf{i} + e^{0.1(10)}\mathbf{j} = 2\mathbf{i} + e\mathbf{j}$

b $\mathbf{a}(t) = 0.1e^{0.1t}\mathbf{j}$

$\mathbf{a}(10) = 0.1e^{0.1(10)}\mathbf{j} = 0.1e\mathbf{j}$

c $\mathbf{r}(t) = \int (2\mathbf{i} + e^{0.1t}\mathbf{j})\,dt = 2t\mathbf{i} + \dfrac{e^{0.1t}}{0.1}\mathbf{j} + \mathbf{c} = 2t\mathbf{i} + 10e^{0.1t}\mathbf{j} + \mathbf{c}$

$\mathbf{r}(0) = 10\mathbf{j}$, so $\mathbf{c} = 0\mathbf{i} + 0\mathbf{j}$

$\mathbf{r}(t) = 2t\mathbf{i} + 10e^{0.1t}\mathbf{j}$

$\mathbf{r}(10) = 2(10)\mathbf{i} + 10e\mathbf{j} = 20\mathbf{i} + 10e\mathbf{j}$

Question 7

$\mathbf{r} = (8t - 12)\mathbf{i} + t^2\mathbf{j}$

a When $t = 3$, $\mathbf{r} = \left[8(3) - 12\right]\mathbf{i} + 3^2\mathbf{j}$

$\mathbf{r} = 12\mathbf{i} + 9\mathbf{j}$ is the position of the particle.

$|\mathbf{r}| = \sqrt{12^2 + 9^2} = 15\text{m}$

When $t = 3$ the particle is 15 metres from the origin.

b $\mathbf{v} = 8\mathbf{i} + 2t\mathbf{j}$

When $t = 3$, $\mathbf{v} = (8\mathbf{i} + 6\mathbf{j})\,\text{m/s}$

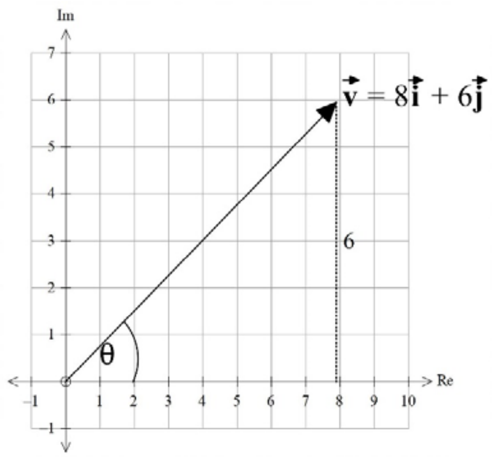
When $t = 3$ the velocity of the particle is $(8\mathbf{i} + 6\mathbf{j})\,\text{m/s}$.

c $|\mathbf{v}| = \sqrt{8^2 + 6^2} = 10\,\text{m/s}$

When $t = 3$, the speed of the particle is 10 m/s.

d $\tan\theta = \dfrac{6}{8}$

$\theta = 37°$ (to the nearest degree)



Question 8

$\mathbf{r} = t^3\mathbf{i} + (2t^2 - 1)\mathbf{j}$

a $\mathbf{v} = 3t^2\mathbf{i} + 4t\mathbf{j}$

When $t = 2$, $\mathbf{v} = 3(2)^2\mathbf{i} + 4(2)\mathbf{j} = 12\mathbf{i} + 8\mathbf{j}$

When $t = 2$, $|\mathbf{v}| = \sqrt{12^2 + 8^2} = 4\sqrt{13}$ m/s

When $t = 2$ the speed of the particle is $4\sqrt{13}$ m/s.

b $\mathbf{a} = 6t\mathbf{i} + 4\mathbf{j}$

When $t = 3$, $\mathbf{a} = 18\mathbf{i} + 4\mathbf{j}$.

When $t = 3$ the acceleration vector is $(18\mathbf{i} + 4\mathbf{j})\,\text{m/s}^2$

c When $t = 2$, $\mathbf{a} = 12\mathbf{i} + 4\mathbf{j}$

$\mathbf{v} \bullet \mathbf{a} = (12\mathbf{i} + 8\mathbf{j}) \bullet (12\mathbf{i} + 4\mathbf{j}) = 12 \times 12 + 8 \times 4 = 176$

d The angle between $\mathbf{v}$ and $\mathbf{a}$ can be found using the scalar product.

$|\mathbf{a}| = \sqrt{12^2 + 4^2} = 4\sqrt{10}$

$$\cos\theta = \frac{176}{4\sqrt{13} \times 4\sqrt{10}} \approx 0.9648$$

$\theta = 15.3°$ (correct to one decimal place)

When $t = 2$, the angle between $\mathbf{v}$ and $\mathbf{a}$ is $15.3°$.

Question 9

Given $\mathbf{v} = 2t\mathbf{i} + (3t^2 - 1)\mathbf{j} - 3\mathbf{k}$, the velocity vector of a particle.

a The initial speed of the particle, when $t = 0$.

When $t = 0$, $\mathbf{v} = -\mathbf{j} - 3\mathbf{k}$

$|\mathbf{v}| = \sqrt{1^2 + 3^2} = \sqrt{10}$ m/s

The initial speed of the particle is $\sqrt{10}$ m/s.

b When $t = 2$, $\mathbf{v} = 4\mathbf{i} + 11\mathbf{j} - 3\mathbf{k}$

$|\mathbf{v}| = \sqrt{4^2 + 11^2 + (-3)^2} = \sqrt{146}$ m/s

The speed of the particle when $t = 2$ is $\sqrt{146}$ m/s.

c $\mathbf{a} = 2\mathbf{i} + 6t\mathbf{j}$

When $t = 2$ the acceleration of the particle is $(2\mathbf{i} + 12\mathbf{j})$ m/s^2.

d $\mathbf{r} = \int \left[2t\mathbf{i} + (3t^2 - 1)\mathbf{j} - 3\mathbf{k} \right] dt = t^2\mathbf{i} + (t^3 - t)\mathbf{j} - 3t\mathbf{k} + \mathbf{c}$

When $t = 2$, the position vector of the particle is $(-4\mathbf{i} + 10\mathbf{j})$ m.

$$t^2\mathbf{i} + (t^3 - t)\mathbf{j} - 3t\mathbf{k} + \mathbf{c} = (-4\mathbf{i} + 10\mathbf{j})$$
$$4\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} + \mathbf{c} = -4\mathbf{i} + 10\mathbf{j}$$
$$\mathbf{c} = -8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{r} = t^2\mathbf{i} + (t^3 - t)\mathbf{j} - 3t\mathbf{k} - 8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$
$$= (t^2 - 8)\mathbf{i} + (t^3 - t + 4)\mathbf{j} + (6 - 3t)\mathbf{k}$$

When $t = 5$, the particle has position vector $(17\mathbf{i} + 124\mathbf{j} - 9\mathbf{k})$ m.

Question 10

$\mathbf{r} = (t^2 - 6t - 16)\mathbf{i} + t^2\mathbf{j}$

a The particle is on the y-axis when the $\mathbf{i}$ component of the position vector must be zero.

$t^2 - 6t - 16 = 0$
$(t-8)(t+2) = 0$
$t = 8, -2$ so 8 is the only valid solution.

The particle is on the y-axis after 8 seconds.

b The particle moves parallel to the y-axis when the $\mathbf{i}$ component of the velocity vector is zero.

$\mathbf{v} = (2t-6)\mathbf{i} + 2t\mathbf{j}$. This occurs when $2t - 6 = 0$, so when $t = 3$.

c When $\mathbf{v} \bullet \mathbf{a} = 0$, the velocity of the particle will be perpendicular to the acceleration of the vehicle.

$\mathbf{a} = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{v} \bullet \mathbf{a} = (2t-6) \times 2 + 2t \times 2$
$8t - 12 = 0$
$t = 1.5$

The velocity of the particle will be perpendicular to the acceleration when $t = 1.5$.

Question 11

$\mathbf{r} = 3\mathbf{i} + 2t\mathbf{j} + (t^2 - 4t + 10)\mathbf{k}$
$\mathbf{v} = 2\mathbf{j} + (2t-4)\mathbf{k}$
$\mathbf{a} = 2\mathbf{k}$

The minimum distance from the particle to the x-y plane is the distance from $\langle 3, 2t, t^2 - 4t + 10 \rangle$, i.e., $\langle 3, 2t, 0 \rangle$.

Hence, the minimum distance will be the minimum value of $t^2 - 4t + 10$.

$t^2 - 4t + 10 = (t-2)^2 + 6$, so the minimum value is when $t = 2$.

$\mathbf{r} = (3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})\,\text{m}$
$\mathbf{v} = 2\mathbf{j}\,\text{m/s}$
$\mathbf{a} = 2\mathbf{k}\,\text{m/s}^2$

Question 12

Given $\mathbf{a} = 2\mathbf{j}$

a $\mathbf{v} = 2t\mathbf{j} + \mathbf{c}$, when $t = 0$ $\mathbf{v} = 2\mathbf{i} - 8\mathbf{j}$

$\mathbf{v} = [2\mathbf{i} + (2t - 8)\mathbf{j}]\,\text{m/s}$

The velocity of the body at time t seconds is $\mathbf{v} = \left[2\mathbf{i} + (2t - 8)\mathbf{j}\right]\text{m/s}$.

b $\mathbf{r} = 2t\mathbf{i} + (t^2 - 8t)\mathbf{j} + \mathbf{c}$ and when $t = 0$, $\mathbf{r} = \mathbf{i} + 20\mathbf{j}$

$\mathbf{c} = \mathbf{i} + 20\mathbf{j}$

$\mathbf{r} = [(2t + 1)\mathbf{i} + (t^2 - 8t + 20)\mathbf{j}]\,\text{m}$

c When $t = 3$, $[(2t + 1)\mathbf{i} + (t^2 - 8t + 20)\mathbf{j}]\,\text{m} = [(2(3) + 1)\mathbf{i} + (3^2 - 8(3) + 20)\mathbf{j}]\,\text{m}$

When $t = 3$, the position vector of the body is $(7\mathbf{i} + 5\mathbf{j})\,\text{m}$

The body will be $\sqrt{7^2 + 5^2} = \sqrt{74}$ m

d When $t = 2$, $\mathbf{v} = [2\mathbf{i} - 4\mathbf{j}]\,\text{m/s}$.

$|\mathbf{v}| = \sqrt{2^2 + 4^2} = 2\sqrt{5}\,\text{m/s}$

e Minimum height occurs when the co-efficient of $\mathbf{j}$ is at its minimum value.

$t^2 - 8t + 20 = (t - 4)^2 + 4$

The minimum height occurs when $t = 4$, the height is 4 m.

f $\mathbf{r} = [(2t + 1)\mathbf{i} + (t^2 - 8t + 20)\mathbf{j}]\,\text{m}$

$x = 2t + 1 \Rightarrow t = \frac{1}{2}(x - 1)$

$y = t^2 - 8t + 20 = \frac{1}{4}(x - 1)^2 - 4(x - 1) + 20$

$4y = x^2 - 2x + 1 - 16x + 16 + 80 = x^2 - 18x + 97$

Question 13

$\mathbf{a} = \cos t\,\mathbf{i} + 2\mathbf{j}$

$\mathbf{v} = \sin t\,\mathbf{i} + 2t\mathbf{j} + \mathbf{c}$

When $t = 0$, the velocity vector is $\mathbf{j}$ so $\mathbf{c} = \mathbf{j}$.

Hence $\mathbf{v} = (\sin t)\,\mathbf{i} + (2t+1)\mathbf{j}$

$$\mathbf{r} = (-\cos t)\mathbf{i} + (t^2 + t)\mathbf{j} + \mathbf{c}$$

When $t = 0$, the position vector is $4\mathbf{i} - 6\mathbf{j}$ so $\mathbf{c} = 3\mathbf{i} - 6\mathbf{j}$.

$$\mathbf{r} = (-\cos t + 3)\mathbf{i} + (t^2 + t - 6)\mathbf{j}$$

a The particle crosses the x-axis when the $\mathbf{j}$ component of the position vector is 0.

$t^2 + t - 6 = 0$

$(t+3)(t-2) = 0$

$t = -3, 2$ so the only valid solution is $t = 2$.

b The particle crosses the y-axis when the $\mathbf{i}$ component of the position vector is 0.

$-\cos t + 3 = 0$

$\cos t = 3$

There are no values of t for which $\cos t = 3$.

So particle does not cross the x-axis.

Question 14

$\mathbf{a} = -4\sin 2t\,\mathbf{i} + 2\mathbf{j} + e^t\mathbf{k}$

$\mathbf{v} = 2\cos 2t\,\mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k} + \mathbf{c}$

When $t = 0$ the particle is at rest, so $\mathbf{v} = 0$.

$\mathbf{v} = 2\,\mathbf{i} + \mathbf{k} + \mathbf{c}$, so $\mathbf{c} = -2\mathbf{i} - \mathbf{k}$

$\mathbf{v} = (2\cos 2t - 2)\mathbf{i} + 2t\mathbf{j} + (e^t - 1)\mathbf{k}$

$\mathbf{r} = (\sin 2t - 2t)\,\mathbf{i} + t^2\mathbf{j} + (e^t - t)\mathbf{k} + \mathbf{c}$

When $t = 0$, the particle is at the origin, so $\mathbf{r} = 0$.

$\mathbf{r} = (\sin 2t - 2t)\,\mathbf{i} + t^2\mathbf{j} + (e^t - t)\mathbf{k} + \mathbf{c}$, so $\mathbf{c} = -\mathbf{k}$

$\mathbf{r} = (\sin 2t - 2t)\,\mathbf{i} + t^2\mathbf{j} + (e^t - t - 1)\mathbf{k}$

When $t = \pi$, $\mathbf{r} = \left[-2\pi\mathbf{i} + \pi^2\mathbf{j} + (e^\pi - \pi - 1)\mathbf{k}\right]\text{m}$

Question 15

$\mathbf{r} = 2\sin 3t\mathbf{i} + 2\cos 3t\mathbf{j}$

a The body crosses the x-axis when the $\mathbf{j}$ component of the position vector is zero.

$$2\cos 3t = 0$$
$$3t = \frac{\pi}{2}$$
$$t = \frac{\pi}{6} \text{ seconds}$$

b Differentiate the expression for the position vector with respect to t to obtain the expression for the velocity at time t, $(6\cos 3t\mathbf{i} - 6\sin 3t\mathbf{j})$ m/s.

Differentiate the expression for the velocity vector with respect to t to obtain the expression for the acceleration at time t, $(-18\sin 3t\mathbf{i} - 18\cos 3t\mathbf{j})$ m/s^2.

c $(6\cos 3t\mathbf{i} - 6\sin 3t\mathbf{j}) \bullet (-18\sin 3t\mathbf{i} - 18\cos 3t\mathbf{j}) = -108(\sin 3t)(\cos 3t) + 108(\sin 3t)(\cos 3t) = 0$

The scalar product of the velocity and acceleration is zero, hence the velocity is perpendicular to the acceleration for all values of t.

Question 16

$\mathbf{a} = 2\sin(0.5t)\mathbf{i} - 2\cos(0.5t)\mathbf{j}$

$\mathbf{v} = -4\cos(0.5t)\mathbf{i} - 4\sin(0.5t)\mathbf{j} + \mathbf{c}$

When $t = 0$, $\mathbf{v} = -4\mathbf{i}$ (given)

Substituting $t = 0$ into the equation we have for $\mathbf{v}$ gives $\mathbf{v} = -4\mathbf{i} + \mathbf{c}$

$\mathbf{c} = 0$

$\mathbf{v} = [-4\cos(0.5t)]\mathbf{i} + [-4\sin(0.5t)]\mathbf{j}$

$\mathbf{r} = [-8\sin(0.5t)]\mathbf{i} + [8\cos(0.5t)]\mathbf{j} + \mathbf{c}$

When $t = 0$, the position vector is $2\mathbf{i} + 8\mathbf{j}$ (given)

Substituting $t = 0$ into the equation we have for $\mathbf{r}$ gives $\mathbf{r} = 8\mathbf{j} + \mathbf{c}$

$\mathbf{c} = 2\mathbf{i}$

$\mathbf{r} = [2 - 8\sin(0.5t)]\mathbf{i} + [8\cos(0.5t)]\mathbf{j}$

When $t = \frac{\pi}{3}$, $\mathbf{r} = -2\mathbf{i} + 4\sqrt{3}\mathbf{j}$

The distance from $2\mathbf{i}$ to $-2\mathbf{i} + 4\sqrt{3}\mathbf{j}$ is:

$$\sqrt{(-4)^2 + (4\sqrt{3})^2} = 8\,\text{m}$$

The object is 8 m from point B.

Exercise 7B

Question 1

From the information given, the velocity $= (u + at)\mathbf{i}$ m/s

By integrating and using the information given, the position vector $= \left(ut + \dfrac{at^2}{2}\right)\mathbf{i}$ m.

Question 2

$\mathbf{a} = -9 \cdot 8\mathbf{j}$
$\mathbf{v} = -9 \cdot 8t\mathbf{j} + \mathbf{c}$
When $t = 0$, $\mathbf{v} = 14\mathbf{i} + 35\mathbf{j}$
$\mathbf{v} = 14\mathbf{i} + (35 - 9.8t)\mathbf{j}$

$\mathbf{r} = 14t\mathbf{i} + (35t - 4.9t^2)\mathbf{j} + \mathbf{c}$, but as its position is O when $t = 0$, $\mathbf{c} = 0$.

The expression for the position vector of the object is $\left[14t\mathbf{i} + (35t - 4.9t^2)\mathbf{j}\right]$ m.

When $t = 5$, the body is at position vector $70\mathbf{i} - 52.5\mathbf{j}$.

The distance from O when $t = 5$ is $\sqrt{50^2 + (-52.5)^2} = 87.5$ m

$x = 14t \Rightarrow t = \dfrac{x}{14}$
$y = 35t - 4.9t^2$
Substituting t gives
$y = 35\left(\dfrac{x}{14}\right) - 4.9\left(\dfrac{x}{14}\right)^2 = \dfrac{5}{2}x - \dfrac{1}{40}x^2$

Question 3

a Acceleration of the particle $= -10\mathbf{j}$ m/s^2

b Initial velocity of the particle $= (80\cos 60°\mathbf{i} + 80\sin 60°\mathbf{j})$ m/s $= \left(40\mathbf{i} + 40\sqrt{3}\mathbf{j}\right)$m/s

c We know that $\mathbf{a} = -10\mathbf{j}$ where $\mathbf{a}$ m/s^2 is the acceleration.

Thus $\mathbf{v} = -10t\mathbf{j} + \mathbf{c}$ where $\mathbf{v}$ m/s is the velocity.

When $t = 0$ $\mathbf{v} = 40\mathbf{i} + 40\sqrt{3}\,\mathbf{j}$ $\therefore \mathbf{c} = 40\mathbf{i} + 40\sqrt{3}\mathbf{j}$

Thus $\mathbf{v} = 40\mathbf{i} + \left(40\sqrt{3} - 10t\right)\mathbf{j}$

The velocity of the particle t seconds after projection is $\mathbf{v} = 40\mathbf{i} + \left(40\sqrt{3} - 10t\right)\mathbf{j}$ m/s.

We know that $\mathbf{v} = 40\mathbf{i} + \left(40\sqrt{3} - 10t\right)\mathbf{j}$

Thus $\mathbf{r} = 40t\,\mathbf{i} + \left(40\sqrt{3}t - 5t^2\right)\mathbf{j} + \mathbf{c}$

When $t = 0$ $\mathbf{r} = 0\mathbf{i} + 0\mathbf{j}$ $\therefore \mathbf{c} = 0\mathbf{i} + 0\mathbf{j}$

The position of the particle t seconds after projection is $\left[40t\mathbf{i} + \left(40\sqrt{3}\,t - 5t^2\right)\mathbf{j}\right]$m .

d The particle is at the horizontal surface when the $\mathbf{j}$ component of the particle is zero.

$40\sqrt{3}\,t - 5t^2 = 0$

$t\left(40\sqrt{3} - 5t\right) = 0$

$t = 0,\ 8\sqrt{3}$

The particle returns to the horizontal surface at $t = 8\sqrt{3}$ s.

e When $t = 8\sqrt{3}$, $\mathbf{r} = 40\left(8\sqrt{3}\right)\mathbf{i} + \left[40\sqrt{3} \times 8\sqrt{3} - 5\left(8\sqrt{3}\right)^2\right]\mathbf{j}$

$\mathbf{r} = 320\sqrt{3}\,\mathbf{i}$ m

The horizontal distance from projection to landing is $320\sqrt{3}$ m.

Question 4

a Acceleration of the particle $\mathbf{a} = -9.8\mathbf{j}\,\text{m/s}^2$

Thus $\mathbf{v} = (-9.8t\mathbf{j} + \mathbf{c})$ m/s

Initial velocity of the particle $\mathbf{v} = (42\cos\theta\mathbf{i} + 42\sin\theta\mathbf{j})$ m/s

$\therefore \mathbf{c} = 42\cos\theta\mathbf{i} + 42\sin\theta\mathbf{j}$

Thus $\mathbf{v} = (42\cos\theta)\mathbf{i} + (42\sin\theta - 9.8t)\mathbf{j}$ m/s

The position vector is then $\mathbf{r} = \left[42t\cos\theta\mathbf{i} + (42t\sin\theta - 4.9t^2)\mathbf{j}\right]$ m

b $42t\cos\theta = 120$ and $42t\sin\theta - 4.9t^2 = 0$

$$t = \frac{120}{42\cos\theta} \text{ and } t(42\sin\theta - 4.9t) = 0$$

$$t = 0,\ t = \frac{42\sin\theta}{4.9}$$

$$\frac{120}{42\cos\theta} = \frac{42\sin\theta}{4.9}$$

$$\sin\theta\cos\theta = \frac{1}{3}$$

$$2\sin\theta\cos\theta = \frac{2}{3}$$

$$\sin 2\theta = \frac{2}{3}$$

$$2\theta = 41.81,\ 138.19$$

$$\theta = 20.9°, 69.1°$$

Question 5

The acceleration of the particle is $-g\mathbf{j}\,\text{m/s}^2$.

Initial velocity of the particle is $(u\cos\theta\mathbf{i}+u\sin\theta\mathbf{j})\,\text{m/s}$

a The velocity of the particle t seconds after projection is $\int(-g\mathbf{j})dt=-gt\mathbf{j}+\mathbf{c}$

When $t=0$, $\mathbf{v}=u\cos\theta°\mathbf{i}+u\sin\theta°$ $\quad\therefore\mathbf{c}=u\cos\theta°\mathbf{i}+u\sin\theta°$

Thus $\quad\mathbf{v}=u\cos\theta°\mathbf{i}+(u\sin\theta°-gt)\mathbf{j}$

b $\mathbf{r}=\int\left[u\cos\theta\mathbf{i}+(u\sin\theta-gt)\mathbf{j}\right]dt=ut\cos\theta\mathbf{i}+\left(ut\sin\theta-\frac{1}{2}gt^2\right)\mathbf{j}+\mathbf{c}$

When $t=0$, $\mathbf{r}=0\quad\therefore\mathbf{c}=0$

$\mathbf{r}=ut\cos\theta°\mathbf{i}+\left(ut\sin\theta°-\frac{1}{2}gt^2\right)\mathbf{j}$

c The particle returns to the horizontal surface when the coefficient of the $\mathbf{j}$ component is 0.

$$-\left(ut\sin\theta°-\frac{1}{2}gt^2\right)=0$$

$$ut\sin\theta°=\frac{1}{2}gt^2$$

$$2ut\sin\theta°=gt^2$$

$$t=\frac{2u\sin\theta°}{g}\text{ seconds}$$

d When $t=\frac{2u\sin\theta°}{g}$, $\mathbf{r}=u\frac{2u\sin\theta°}{g}\cos\theta°\mathbf{i}-0\mathbf{j}$

$2\sin\theta\cos\theta=\sin2\theta$

so $\mathbf{r}=\frac{u^2\sin2\theta°}{g}\mathbf{i}$ metres

e The maximum value of $\sin2\theta°$ is 1.

This would give the maximum horizontal distance from O to the point of landing back on the horizontal plane.

$$\sin2\theta°=1$$

$$2\theta°=90°$$

$$\theta=45$$

Question 6

Given $\mathbf{r}(t) = 2\cos(0.5t)\mathbf{i} + 2\sin(0.5t)\mathbf{j}$

a $\mathbf{v}(t) = [-\sin(0.5t)\mathbf{i} + \cos(0.5t)\mathbf{j}]\text{ m/s}$

$\mathbf{a}(t) = [-0.5\cos(0.5t)\mathbf{i} - 0.5\sin(0.5t)\mathbf{j}]\text{ m/s}^2$

b $|\mathbf{v}(t)| = \sqrt{[-\sin(0.5t)]^2 + [\cos(0.5t)]^2} = 1$

c $\mathbf{v} \bullet \mathbf{a} = 0.5\sin(0.5t)\cos(0.5t) - 0.5\sin(0.5t)\cos(0.5t) = 0$

Velocity is always perpendicular to acceleration.

d $\mathbf{r}(t) = 2\cos(0.5t)\mathbf{i} + 2\sin(0.5t)\mathbf{j}$

$\mathbf{a}(t) = -0.5\cos(0.5t)\mathbf{i} - 0.5\sin(0.5t)\mathbf{j}$

$\mathbf{a}(t) = -\frac{1}{4}\mathbf{r}(t)$

Given $\mathbf{a}(t) = -k\mathbf{r}(t)$

$k = \frac{1}{4}$

e With $k > 0$, the acceleration is always directed towards the centre of the circle, (0, 0)

Question 7

Given $\mathbf{v}(t) = -\frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)\mathbf{i} - \frac{5\pi}{2}\sin\left(\frac{\pi}{2}t\right)\mathbf{j}$

a $\mathbf{r}(t) = \int\left[-\frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)\mathbf{i} - \frac{5\pi}{2}\sin\left(\frac{\pi}{2}t\right)\mathbf{j}\right]dt$

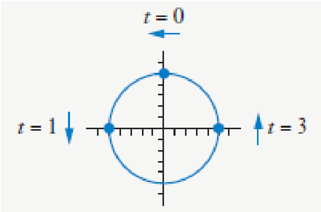
$\mathbf{r}(t) = -5\sin\left(\frac{\pi}{2}t\right)\mathbf{i} + 5\cos\left(\frac{\pi}{2}t\right)\mathbf{j} + \mathbf{c}$

$\mathbf{r}(0) = 5\mathbf{j}$ so $\mathbf{c} = 0$

$\mathbf{r}(t) = -5\sin\left(\frac{\pi}{2}t\right)\mathbf{i} + 5\cos\left(\frac{\pi}{2}t\right)\mathbf{j}$

b $\mathbf{r}(3) = -5\sin\left(\frac{3\pi}{2}\right)\mathbf{i} + 5\cos\left(\frac{3\pi}{2}\right)\mathbf{j} = -5(-1)\mathbf{i} + 5(0)\mathbf{j} = 5\mathbf{i}$

c



d

$$\int_0^3\left[-\frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)\mathbf{i}-\frac{5\pi}{2}\sin\left(\frac{\pi}{2}t\right)\mathbf{j}\right]dt=\left[-5\sin\left(\frac{\pi}{2}t\right)\mathbf{i}+5\cos\left(\frac{\pi}{2}t\right)\mathbf{j}\right]_0^3$$

$$=-5\sin\left(\frac{3\pi}{2}\right)\mathbf{i}+5\cos\left(\frac{3\pi}{2}\right)\mathbf{j}-\left(-5\sin 0\mathbf{i}+5\cos 0\mathbf{j}\right)$$

$$=5\mathbf{i}+0\mathbf{j}-(0\mathbf{i}+5\mathbf{j})$$

$$=(5\mathbf{i}-5\mathbf{j})\,\text{m}$$

This is the vector from $\mathbf{r}(0)$ to $\mathbf{r}(3)$.

It is the displacement vector for $t=0$ to $t=3$.

$$\left|\int_0^3\mathbf{v}(t)\text{d}t\right|=|5\mathbf{i}-5\mathbf{j}|=\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2}$$

This is the magnitude of the displacement from $t=0$ to $t=3$.

$$\int_0^3|\mathbf{v}(t)|\,dt=\int_0^3\left|-\frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)\mathbf{i}-\frac{5\pi}{2}\sin\left(\frac{\pi}{2}t\right)\mathbf{j}\right|dt$$

$$=\int_0^3\sqrt{\left(-\frac{5\pi}{2}\cos\frac{\pi}{2}t\right)^2+\left(-\frac{5\pi}{2}\sin\frac{\pi}{2}t\right)^2}\,dt$$

$$=\int_0^3\sqrt{\left(\frac{25\pi^2}{4}\cos^2\frac{\pi}{2}t\right)+\left(\frac{25^2\pi^2}{4}\sin^2\frac{\pi}{2}t\right)}dt$$

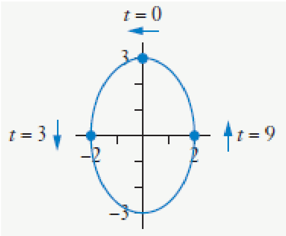
$$=\int_0^3\sqrt{\frac{25\pi^2}{4}\left(\cos^2\frac{\pi}{2}t+\sin^2\frac{\pi}{2}t\right)}dt=\int_0^3\sqrt{\frac{25\pi^2}{4}}dt$$

$$=\int_0^3\frac{5\pi}{2}dt=\left[\frac{5\pi}{2}t\right]_0^3=\frac{15\pi}{2}$$

This is the distance travelled from $t=0$ to $t=3$, i.e. three quarters of the circumference.

Question 8

a



b

$$\begin{cases} x = -2\sin\left(\frac{\pi}{6}t\right) \\ y = 3\cos\left(\frac{\pi}{6}t\right) \end{cases} \Rightarrow \begin{cases} \frac{x}{-2} = \sin\left(\frac{\pi}{6}t\right) \\ \frac{y}{3} = \cos\left(\frac{\pi}{6}t\right) \end{cases}$$

$$\left(\frac{x}{-2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left[\sin\left(\frac{\pi}{6}t\right)\right]^2 + \left[\cos\left(\frac{\pi}{6}t\right)\right]^2$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{36x^2}{4} + \frac{36y^2}{9} = 36$$

$$9x^2 + 4y^2 = 36$$

c

$$\mathbf{r}(t) = -2\sin\left(\frac{\pi}{6}t\right)\mathbf{i} + 3\cos\left(\frac{\pi}{6}t\right)\mathbf{j}$$

$$\mathbf{v}(t) = -\frac{\pi}{3}\cos\left(\frac{\pi}{6}t\right)\mathbf{i} - \frac{\pi}{2}\sin\left(\frac{\pi}{6}t\right)\mathbf{j}$$

$$\mathbf{v}(8) = -\frac{\pi}{3}\cos\left(\frac{4\pi}{3}\right)\mathbf{i} - \frac{\pi}{2}\sin\left(\frac{4\pi}{3}\right)\mathbf{j} = \frac{\pi}{6}\mathbf{i} + \frac{\sqrt{3}\pi}{4}\mathbf{j}$$

$$\tan\theta = \frac{\frac{\sqrt{3}\pi}{4}}{\frac{\pi}{6}} = \frac{3\sqrt{3}}{2}$$

$$\theta = 1.20\,\text{radians}$$

d

$$\mathbf{r}(t) = -2\sin\left(\frac{\pi}{6}t\right)\mathbf{i} + 3\cos\left(\frac{\pi}{6}t\right)\mathbf{j}$$

$$\mathbf{v}(t) = -\frac{\pi}{3}\cos\left(\frac{\pi}{6}t\right)\mathbf{i} - \frac{\pi}{2}\sin\left(\frac{\pi}{6}t\right)\mathbf{j}$$

$$\mathbf{a}(t) = \frac{\pi^2}{18}\sin\left(\frac{\pi}{6}t\right)\mathbf{i} - \frac{\pi^2}{12}\cos\left(\frac{\pi}{6}t\right)\mathbf{j} = -\frac{\pi^2}{36}\left[-2\sin\left(\frac{\pi}{6}t\right)\mathbf{i} + 3\cos\left(\frac{\pi}{6}t\right)\mathbf{j}\right] = -\frac{\pi^2}{36}\mathbf{r}(t)$$

$$k = -\frac{\pi^2}{36}$$

Acceleration is always towards (0, 0).

Question 9

$$\mathbf{a}(t) = (-9.8\sin 30°\mathbf{i} - 9.8\cos 30°\mathbf{j})\text{ m/s}^2 = \left(-\frac{49}{10}\mathbf{i} - \frac{49\sqrt{3}}{10}\mathbf{j}\right)$$

$$\mathbf{v}(t) = -\frac{49}{10}t\,\mathbf{i} - \frac{49\sqrt{3}}{10}t\,\mathbf{j} + \mathbf{c}$$

$$\mathbf{v}(0) = 49\cos 30°\,\mathbf{i} + 49\cos 30°\,\mathbf{j}$$

$$\mathbf{c} = 49\cos 30°\,\mathbf{i} + 49\sin 30°\,\mathbf{j}$$

$$\mathbf{v}(t) = -\frac{49}{10}t\,\mathbf{i} - \frac{49\sqrt{3}}{10}t\,\mathbf{j} + 49\cos 30°\,\mathbf{i} + 49\sin 30°\,\mathbf{j}$$

$$= \left(\frac{49\sqrt{3}}{2} - \frac{49}{10}t\right)\mathbf{i} + \left(\frac{49}{2} - \frac{49\sqrt{3}}{10}t\right)\mathbf{j}$$

$$\mathbf{r}(t) = \left(\frac{49\sqrt{3}}{2}t - \frac{49}{20}t^2\right)\mathbf{i} + \left(\frac{49}{2}t - \frac{49\sqrt{3}}{20}t^2\right)\mathbf{j} + \mathbf{c}$$

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{r}(t) = \left(\frac{49\sqrt{3}}{2}t - \frac{49}{20}t^2\right)\mathbf{i} + \left(\frac{49}{2}t - \frac{49\sqrt{3}}{20}t^2\right)\mathbf{j}$$

$$= \frac{49}{20}t\left(10\sqrt{3} - t\right)\mathbf{i} + \frac{49}{20}t\left(10 - \sqrt{3}t\right)\mathbf{j}$$

The particle will hit the plane when the co-efficient of the **j** component is 0.

$$\frac{49}{2}t - \frac{49\sqrt{3}}{20}t^2 = 0$$

$$t\left(\frac{49}{2} - \frac{49\sqrt{3}}{20}t\right) = 0$$

$$t = 0,\ \frac{10\sqrt{3}}{3}$$

So the particle will hit the plane after $\frac{10\sqrt{3}}{3}$ seconds.

Question 10

a $\mathbf{v}(t) = (1-\cos t)\mathbf{i} + \sin t\,\mathbf{j}$

$\mathbf{r}(t) = (t-\sin t)\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{c}$

When $t = 0$, point P lies at the origin.

$\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j} + \mathbf{c}$

$\mathbf{c} = \mathbf{j}$

$\mathbf{r}(t) = (t-\sin t)\mathbf{i} + (1-\cos t)\,\mathbf{j}$

b $\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$

$\mathbf{r}(\pi) = (\pi - \sin\pi)\mathbf{i} + (1-\cos\pi)\,\mathbf{j} = \pi\mathbf{i} + 2\mathbf{j}$

Looking at the value of the vertical components, the diameter is 2 m.

c **i** $\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$

$\mathbf{v}(0) = 0\mathbf{i} + 0\,\mathbf{j}$

ii $\mathbf{r}\left(\dfrac{\pi}{2}\right) = \left(\dfrac{\pi}{2} - 1\right)\mathbf{i} + \mathbf{j}$

$\mathbf{v}\left(\dfrac{\pi}{2}\right) = \mathbf{i} + \mathbf{j}$

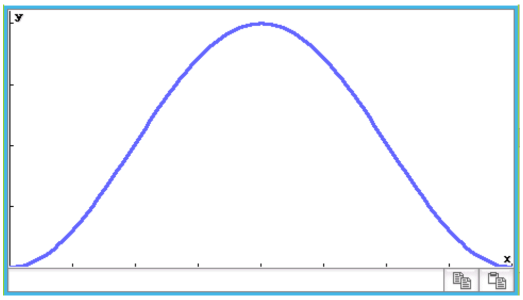
iii $\mathbf{r}(\pi) = \pi\mathbf{i} + 2\,\mathbf{j}$

$\mathbf{v}(t) = 2\mathbf{i} + 0\mathbf{j}$

iv $\mathbf{r}\left(\dfrac{3\pi}{2}\right) = \left(\dfrac{3\pi}{2} + 1\right)\mathbf{i} + \mathbf{j}$

$\mathbf{v}\left(\dfrac{3\pi}{2}\right) = \mathbf{i} - \mathbf{j}$

d



Question 1

Given $-\sqrt{3}+i$

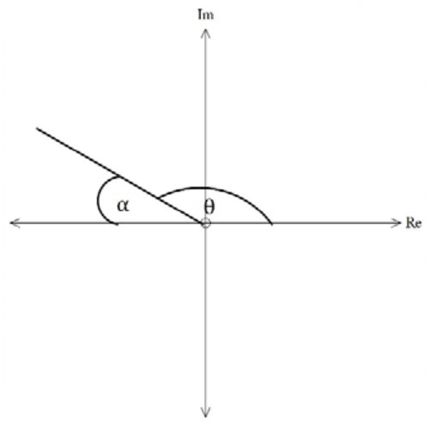
$\mathbf{r}=\sqrt{\sqrt{3}^2+1^2}=\sqrt{4}=2$

$\tan\alpha=\dfrac{1}{\sqrt{3}}$

$\alpha=\dfrac{\pi}{6}$

$\theta=\pi-\dfrac{\pi}{6}=\dfrac{5\pi}{6}$

$-\sqrt{3}+i=2\text{cis}\left(\dfrac{5\pi}{6}\right)$



Question 2

$6\cos\left(\dfrac{3\pi}{4}\right)=-3\sqrt{2},\qquad 6\sin\left(\dfrac{3\pi}{4}\right)=3\sqrt{2}$

$6\,\text{cis}\left(\dfrac{3\pi}{4}\right)=-3\sqrt{2}+3\sqrt{2}\,i$

Question 3

$$f(x)=\begin{cases} x\div 4 & \text{for} \quad x\le 0 \\ x^2 & \text{for} \quad 0<x<3 \\ 3x & \text{for} \quad x\ge 3 \end{cases}$$

$$f^{-1}(x)=\begin{cases} 4x & \text{for} \quad x\le 0 \\ \sqrt{x} & \text{for} \quad 0<x<9 \\ x\div 3 & \text{for} \quad x\ge 9 \end{cases}$$

Question 4

All of the equations $f(x), g(x)$ and $h(x)$ match the graph.

Question 5

Given $f(x)=3+\sqrt{x+1}$

$$x \xrightarrow{+1} x+1 \quad \xrightarrow{\sqrt{\ }} \sqrt{x+1} \xrightarrow{+3} 3+\sqrt{x+1}$$
$$(x-3)^2-1 \xleftarrow{-1} (x-3)^2 \xleftarrow{(x)^2} \quad x-3 \xleftarrow{-3} x$$

$f^{-1}(x)=(x-3)^2-1$

Domain: $\{x \in \mathbb{R} : x \geq 3\}$

Range: $\{y \in \mathbb{R} : y \geq -1\}$

Question 6

a $\mathbf{p}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$

b $\mathbf{q}=-2\mathbf{i}+4\mathbf{j}+3\mathbf{k}$

c $\cos\theta=\dfrac{\mathbf{p}\bullet\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}=\dfrac{-4+12-6}{\sqrt{17}\sqrt{29}}$

$\theta \approx 84.832 \approx 85°$

The angle between $\mathbf{p}$ and $\mathbf{q}$ is approximately 85°.

d The x-axis has vector $\mathbf{i}+0\mathbf{j}+0\mathbf{k}$

$\cos\theta=\dfrac{2}{\sqrt{17}\sqrt{1}}$

$\theta \approx 60.983 \approx 61°$

The angle between $\mathbf{p}$ and the x-axis has approximately 61°.

e The y-axis has vector $0\mathbf{i}+\mathbf{j}+0\mathbf{k}$

$\cos\theta=\dfrac{4}{\sqrt{29}\sqrt{1}}$

$\theta \approx 42.031 \approx 42°$

The angle between $\mathbf{q}$ and the y-axis has approximately 42°.

Question 7

The resultant of $\mathbf{i}+5\mathbf{j}-4\mathbf{k}$ and $\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ is $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

$|2\mathbf{i}+2\mathbf{j}-\mathbf{k}|=\sqrt{2^2+2^2+(-1)^2}=\sqrt{9}=3$

$\frac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$

A unit vector parallel to the resultant of $\mathbf{i}+5\mathbf{j}-4\mathbf{k}$ and $\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ is $\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$.

Question 8

$\mathbf{a}=\begin{pmatrix}2\\3\\-2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}3\\-2\\2\end{pmatrix}$, $\quad \mathbf{a}\times\mathbf{b}=\begin{pmatrix}3\times 2-(-2)(-2)\\-2\times 2+3(-2)\\2\times(-2)-3(3)\end{pmatrix}=\begin{pmatrix}2\\-10\\-13\end{pmatrix}$

Question 9

$g(f(x))=\sqrt{4-5\sqrt{x}}$

The maximum value of $5\sqrt{x}$ is 4, $x=\frac{16}{25}=0.64$

Domain: $\{x\in\mathbb{R}:0\le x\le 0.64\}$

Minimum value of $g(f(x))$ occurs when $5\sqrt{x}=4$, $g(f(x))=0$

Maximum value of $g(f(x))$ occurs when $x=0$, $g(f(x))=2$

Range: $\{y\in\mathbb{R}:0\le y\le 2\}$

Question 10

$\mathbf{r}=(6t+1)\mathbf{i}+(t^3+t^2+8t)\mathbf{j}$

When $t=0$, $\mathbf{r}=\mathbf{i}$.

$\mathbf{v}=6\mathbf{i}+(3t^2+2t+8)\mathbf{j}$

When $t=0$, $\mathbf{v}=6\mathbf{i}+8\mathbf{j}$.

$|\mathbf{v}|=\sqrt{6^2+8^2}=\sqrt{100}=10$

$\mathbf{a}=(6t+2)\mathbf{j}$

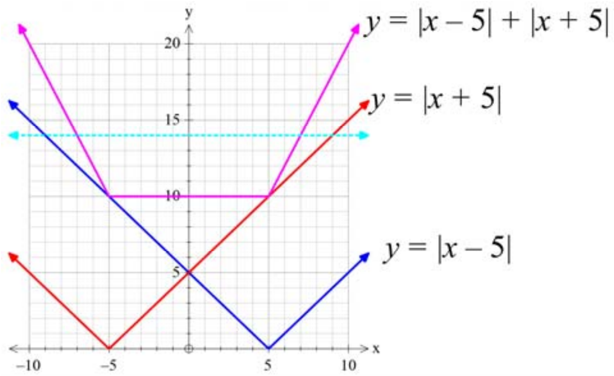
When $t=0$, $\mathbf{a}=2\mathbf{j}$.

Initial velocity is $(6\mathbf{i}+8\mathbf{j})$ m/s

Initial speed is 10 m/s

Initial acceleration is $2\mathbf{j}$ m/s^2

Question 11



$|x-5|+|x+5| \le 14$ for $-7 \le x \le 7$.

Question 12

$z = a + ib, \qquad \bar{z} = a - ib$

a $\quad z + \bar{z} = a + ib + a - ib = 2a$

b $\quad z - \bar{z} = a + ib - (a - ib) = 2ib$

c $\quad z\bar{z} = (a+ib)(a-ib) = a^2 - iab + iab - i^2b^2 = a^2 - (-1)b^2 = a^2 + b^2$

d $\quad \dfrac{z}{\bar{z}} = \dfrac{a+ib}{a-ib} \times \dfrac{a+ib}{a+ib} = \dfrac{a^2 + 2iab + i^2b^2}{a^2+b^2} = \dfrac{a^2 + 2iab - b^2}{a^2+b^2} = \dfrac{a^2 - b^2}{a^2+b^2} + i\dfrac{2ab}{a^2+b^2}$

Question 13

Point P has position vector $\begin{pmatrix} 1 - \frac{4}{5} \times 5 \\ 6 - \frac{4}{5} \times 5 \\ -7 + \frac{4}{5} \times 10 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$.

Question 14

$f \circ g(x) = \sqrt{x-9}^{\,2} = x - 9$

Domain: $\{x \in \mathbb{R} : x \ge 9\}$, Range: $\{y \in \mathbb{R} : y \ge 0\}$

$g \circ f(x) = \sqrt{x^2 - 9}$

Domain: $\{x \in \mathbb{R} : |x| \ge 3\}$, Range: $\{y \in \mathbb{R} : y \ge 0\}$

Question 15

$f \circ g(x) = \sqrt{9-x}^{\,2} = 9-x$

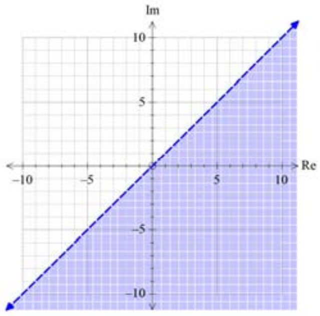
Domain: $\{x \in \mathbb{R} : x \leq 9\}$, Range: $\{y \in \mathbb{R} : y \geq 0\}$

$g \circ f(x) = \sqrt{9-x^2}$

Domain: $\{x \in \mathbb{R} : -3 \leq x \leq 3\}$, Range: $\{y \in \mathbb{R} : 0 \leq y \leq 3\}$

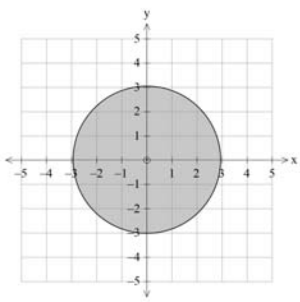
Question 16

a $\operatorname{Re} z > \operatorname{Im} z$

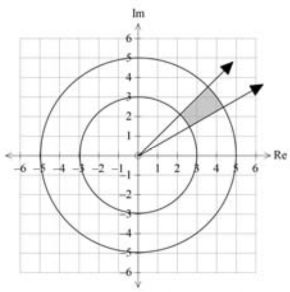


(note the use of the dashed line to imply the line itself is not included)

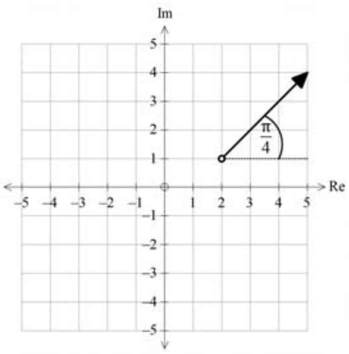
b $|z| \leq 3$



c Both $3 \leq |z| \leq 5$ and $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$



d $\arg[z-(2+i)] = \frac{\pi}{4}$



(note the horizontal line is dashed to imply that the line itself is not included)

Question 17

$$|z-1| = 2|z-i|$$
$$|x+iy-1| = 2|x+iy-i| = 2|x+i(y-1)|$$
$$(x-1)^2 + y^2 = 2^2\left[x^2 + (y-1)^2\right]$$
$$x^2 - 2x + 1 + y^2 = 4x^2 + 4y^2 - 8y + 4$$
$$3x^2 + 2x + 3y^2 - 8y = -3$$
$$x^2 + \frac{2}{3}x + y^2 - \frac{8}{3}y = -1$$
$$\left(x+\frac{1}{3}\right)^2 - \frac{1}{9} + \left(y-\frac{4}{3}\right)^2 - \frac{16}{9} = -\frac{9}{9}$$
$$\left(x+\frac{1}{3}\right)^2 + \left(y-\frac{4}{3}\right)^2 = \frac{8}{9}$$

The points form a circle with centre $\left(-\frac{1}{3}, \frac{4}{3}\right)$ and radius $\frac{2\sqrt{2}}{3}$.

Question 18

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} -8 \\ 4 \\ 1 \end{pmatrix} = 0$$
$$-8a + 4b + c = 0$$
$$\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{a^2 + b^2 + c^2}$$

Unit vector perpendicular to $\begin{pmatrix} -8 \\ 4 \\ 1 \end{pmatrix}$ is $\frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, with $-8a + 4b + c = 0$.

There are many possible answers, for example $\frac{1}{\sqrt{17}}\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}, \frac{1}{9}\begin{pmatrix} 1 \\ 4 \\ -8 \end{pmatrix}$.

Question 19

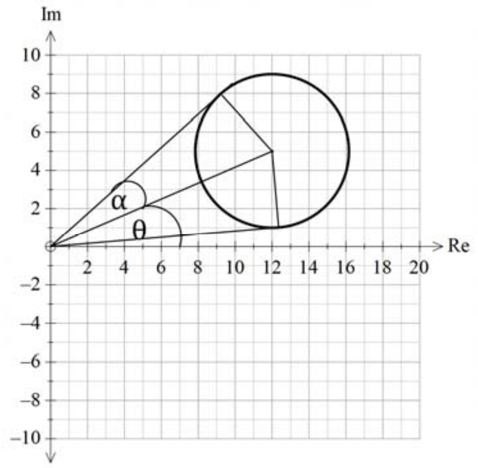
a The minimum possible value of $\text{Im}(z)$ is 1.

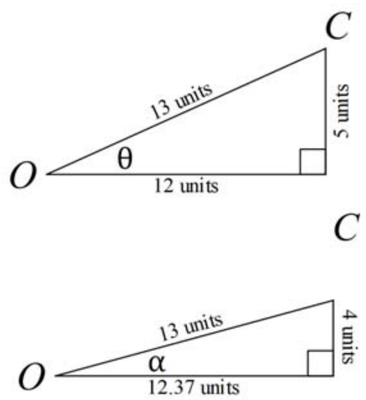
b The maximum possible value of $|\text{Re}(z)|$ is 16.

c Distance from origin to centre of circle is $\sqrt{12^2+5^2}=13\,\text{units}$ and radius of the circle is 4.

Maximum value of $|z|$ is 13 + 4 = 17 units.

d Minimum value of $|z|$ is 13 – 4 = 9 units.





$$\tan\theta = \frac{5}{12}$$
$$\theta = 0.3948 \text{ radians}$$
$$\tan\alpha = \frac{4}{12.37}$$
$$\alpha = 0.3128 \text{ radians}$$

e 0.3948 – 0.3128 = 0.082 radians.

The minimum possible value of $\arg(z)$ is 0.082 radians.

f 0.3948 + 0.3128 = 0.708 radians.

The maximum possible value of $\arg(z)$ is 0.708 radians.

Question 20

The lines will intersect if $L_1 = L_2$, so when:

$$\begin{aligned} 3+4\lambda &= 2+3\mu \\ 2+\lambda &= 1+\mu \\ -1+3\lambda &= 1+2\mu \end{aligned}$$

There is no solution for this system of equations, therefore L_1 and L_2 do not intersect.

Question 21

$a=3, \quad b=5, \quad c=3$

$|\mathbf{r}|=3$ intersects with $|\mathbf{r}|=3\theta$

$3=3\theta$

$\theta=1$

The point of intersection is (3, 1).

$|\mathbf{r}|=5$ intersects with $|\mathbf{r}|=3\theta$

$5=3\theta$

$\theta=\frac{5}{3}$

The point of intersection is $\left(5, \frac{5}{3}\right)$.

Question 22

Line L , with equation, $\mathbf{r}=\begin{pmatrix}-1\\-10\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}$ meets plane with equation $\mathbf{r}\bullet\begin{pmatrix}2\\4\\-1\end{pmatrix}=5$, where

$$\begin{pmatrix}-1+2\lambda\\-10+3\lambda\\4-\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\4\\-1\end{pmatrix}=5.$$

$$\begin{aligned} -2+4\lambda-40+12\lambda-4+\lambda &= 5 \\ 17\lambda-46 &= 5 \\ \lambda &= 3 \end{aligned}$$

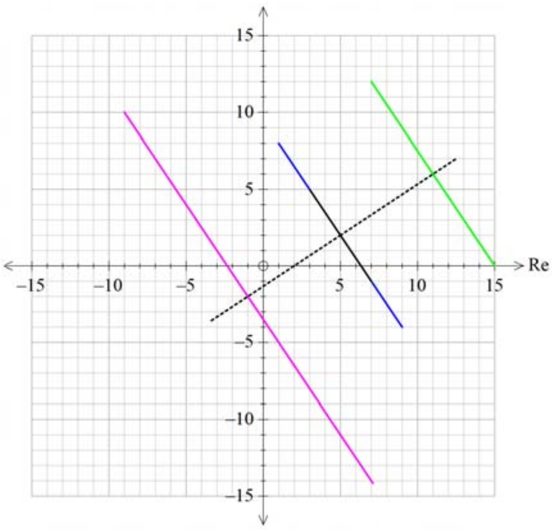
$$\mathbf{r}=\begin{pmatrix}-1\\-10\\4\end{pmatrix}+3\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}5\\-1\\1\end{pmatrix}$$

Question 23

From the blue lines on the Argand diagram it can be seen that $a = 9,\ b = -4$.

From the green line it can be seen that $c = 7,\ d = 12$.

From the pink line it can be seen that $e = 9,\ f = -10$.



Question 24

$$\begin{aligned} 2\lambda + \mu + 2\eta &= -6 \\ 3\lambda + 4\mu + 0\eta &= 5 \\ \lambda - \mu + 0\eta &= -3 \end{aligned}$$

Solving gives $\lambda = -1,\ \mu = 2,\ \eta = -3$.

$$\begin{pmatrix} -6 \\ 5 \\ -3 \end{pmatrix} = -\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$$

Question 25

Point *F* has position vector $\left(\frac{-1+3}{2}\right)\mathbf{i}+\left(\frac{6+0}{2}\right)\mathbf{j}+\left(\frac{-8+4}{2}\right)\mathbf{k}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}$

F is the centre of the sphere.

$\left|\overrightarrow{BF}\right|=\sqrt{2^2+(-3)^2+6^2}=7$, so the radius of the sphere is 7.

The magnitude of $\overrightarrow{CF}$ must be 7 for point *C* to lie on the surface of the sphere.

$\sqrt{6^2+(-3)^2+(c+2)^2}=7$

$c=0$

The magnitude of $\overrightarrow{DF}$ must be 7 for point *D* to lie on the surface of the sphere.

$\sqrt{(-2)^2+(d-3)^2+(-3)^2}=7$

$d=9$

The magnitude of $\overrightarrow{EF}$ must be 7 for point *E* to lie on the surface of the sphere.

$\sqrt{3^2+(-2)^2+(e+2)^2}=7$

$e=4$

Question 26

a $\begin{pmatrix}1\\0\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\3\\-1\end{pmatrix}=4$

$4=\sqrt{1^2+(-2)^2}\times\sqrt{2^2+3^2+(-1)^2}\times\cos\theta=\sqrt{70}\cos\theta$

$\theta\approx 1.07\text{ radians}$

b From the equations for L_1 and L_2, if the two lines were to meet it would follow that:

$-2+\lambda=5+2\mu$

$8-2\lambda=-3-\mu$

Solving gives $\mu=-1,\ \lambda=5$.

The point of intersection of the two lines has position vector $\mathbf{r}=3\mathbf{i}+3\mathbf{j}-2\mathbf{k}$.

c $\mathbf{r}\bullet\mathbf{n}=\mathbf{a}\bullet\mathbf{n}$

$\mathbf{r}\bullet\left\{(3-2)\mathbf{i}+\left[1-(-1)\right]\mathbf{j}+(1-3)\mathbf{k}\right\}=(3\mathbf{i}+3\mathbf{j}-2\mathbf{k})\bullet(\mathbf{i}+2\mathbf{j}-2\mathbf{k})$

$\mathbf{r}\bullet(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=13$

Question 27

Assume that L_1 intersects with L_2.

It follows that:

$2+4\lambda=8+2\mu$

$3-2\lambda=\mu$

Solving gives $\lambda=\frac{3}{2},\ \mu=0$.

We also know that $-1+3\lambda=1+2\mu$, but

$-1+3\left(\frac{3}{2}\right)\neq 1+2(0)$, therefore L_1 and L_2 do not intersect.

Question 28

$$\begin{aligned}\cos 4\theta &= 2\cos^2 2\theta-1\\ &=2(2\cos^2\theta-1)^2-1\\ &=2(4\cos^4\theta-4\cos^2\theta+1)-1\\ &=8\cos^4\theta-8\cos^2\theta+1\end{aligned}$$

Question 29

Given $z_1=a+b\text{i}$

$z_2=-b+a\text{i}$, $z_3=-a-b\text{i}$, $z_4=b-a\text{i}$

Question 30

a

$$x+2y+z=7 \quad \text{Eq}^{\text{n}}① $$
$$x+3y+2z=11 \quad \text{Eq}^{\text{n}}② $$
$$2x+5y+5z=26 \quad \text{Eq}^{\text{n}}③ $$

$$\text{Eq}^{\text{n}}① \qquad x+2y+z=7$$
$$\text{Eq}^{\text{n}}②-\text{Eq}^{\text{n}}① \qquad y+z=4$$
$$\text{Eq}^{\text{n}}③-2\text{Eq}^{\text{n}}① \qquad y+3z=12$$

$$\text{Eq}^{\text{n}}① \qquad x+2y+z=7$$
$$\text{Eq}^{\text{n}}②' \qquad y+z=4$$
$$\text{Eq}^{\text{n}}③-\text{Eq}^{\text{n}}② \qquad 2z=8$$

$z=4$

$y=0$

$x=3$

b

$$2x+3y+5z=4 \quad \text{Eq}^{\text{n}}① $$
$$x+2z=1 \quad \text{Eq}^{\text{n}}② $$
$$3x+y+7z=3 \quad \text{Eq}^{\text{n}}③ $$

$$\text{Eq}^{\text{n}}①-2\text{Eq}^{\text{n}}② \qquad 3y+z=2$$
$$\text{Eq}^{\text{n}}② \qquad x+2z=1$$
$$\text{Eq}^{\text{n}}③-3\text{Eq}^{\text{n}}② \qquad y+z=0$$

$$\text{Eq}^{\text{n}}①-3\text{Eq}^{\text{n}}③ \qquad -2z=2$$
$$\text{Eq}^{\text{n}}②' \qquad x+2z=1$$
$$\text{Eq}^{\text{n}}③' \qquad y+z=0$$

$z=-1$

$y=1$

$x=3$

Question 31

$$x=2\sec\left(t-\frac{\pi}{2}\right), \qquad y=4\tan\left(t-\frac{\pi}{2}\right)$$

$$\left(\frac{x}{2}\right)^2-\left(\frac{y}{4}\right)^2=\sec^2\left(t-\frac{\pi}{2}\right)-\tan^2\left(t-\frac{\pi}{2}\right)=\frac{1-\sin^2\left(t-\frac{\pi}{2}\right)}{\cos^2\left(t-\frac{\pi}{2}\right)}$$

$$=\frac{1-\left(1-\cos^2\left(t-\frac{\pi}{2}\right)\right)}{\cos^2\left(t-\frac{\pi}{2}\right)}=\frac{\cos^2\left(t-\frac{\pi}{2}\right)}{\cos^2\left(t-\frac{\pi}{2}\right)}$$

$$\frac{x^2}{4}-\frac{y^2}{16}=1$$

$4x^2-y^2=16$ (for $x\geq 2$).

Question 32

$180 - 15 \times 3\lambda = 0$

$\lambda = 4$

$\mathbf{r} = 600\mathbf{i} + 240\mathbf{j} + 180\mathbf{k} - t(40\mathbf{i} + 16\mathbf{j} + 12\mathbf{k})$

$\mathbf{v} = \left[-40\mathbf{i} - 16\mathbf{j} - 12\mathbf{k}\right]\text{m/s}$

$\left|\mathbf{r}_0 - \mathbf{r}_{15}\right| = \sqrt{600^2 + 240^2 + 180^2} = \sqrt{450\,000} \approx 670.82$

The glider travels approximately 670 m.

Question 33

$x + y + (p-3)z = 1$ Eq^n ①

$2x + 4y + 4z = 3$ Eq^n ②

$-x + y + (7-2p)z = q$ Eq^n ③

Eq^n ① $x + y + (p-3)z = 1$

Eq^n ②–2Eq^n ① $2y + (10-2p)z = 1$

Eq^n ③+Eq^n ① $2y + (4-p)z = q+1$

Eq^n ① $x + y + (p-3)z = 1$

Eq^n ②' $2y + (10-2p)z = 1$

Eq^n ③–Eq^n ②' $(p-6)z = q$

If $p = 6$, $0 = q$.

There is no value of q that will give a unique solution for this system of linear equations.

a **i** If $p = 6$ and $q = 0$ there are infinite solutions for this system of linear equations.

ii If $p = 6$ and $q \neq 0$ there is no solution for this system of linear equations.

b If $p = 5$, $z = -q$, $y = \frac{1}{2}$ and $x = 2q + \frac{1}{2}$.

c If $p = 6$ and $q = 0$, when $x = 1$:

$y + 3z = 0$

$2y - 2z = 1$

$z = -\frac{1}{8}, \quad y = \frac{3}{8}, \quad x = 1$

Question 34

a

$\dot{\mathbf{r}} = 4\cos 2t\,\mathbf{i} + 3\mathbf{j}$

$\mathbf{r} = 2\sin 2t\,\mathbf{i} + 3t\mathbf{j} + \mathbf{c}$

When $t = 0$, $\mathbf{r} = 2\mathbf{i} - \mathbf{j}$

$\mathbf{c} = 2\mathbf{i} - \mathbf{j}$

$\mathbf{r} = (2\sin 2t + 2)\,\mathbf{i} + (3t - 1)\,\mathbf{j}$

When $t = \pi$, $\mathbf{r} = 2\mathbf{i} + (3\pi - 1)\,\mathbf{j}$

b

$\dot{\mathbf{r}} = 4\cos 2t\,\mathbf{i} + 3\mathbf{j}$

$\ddot{\mathbf{r}} = -8\sin 2t\,\mathbf{i} + 0\mathbf{j}$

$\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}} = 0$

$$(4\cos 2t\,\mathbf{i} + 3\mathbf{j}) \bullet (-8\sin 2t\mathbf{i} + 0\mathbf{j}) = 0$$
$$-32\sin 2t\cos 2t = 0$$
$$-16\sin 4t = 0$$

$4t = \pi,\ t > 0 \ \Rightarrow \ t = \dfrac{\pi}{4},\ t > 0$

When $t = \dfrac{\pi}{4}$:

The position vector is $\mathbf{r} = \left[4\mathbf{i} + \left(\dfrac{3\pi}{4} - 1\right)\mathbf{j}\right]$m

$|\dot{\mathbf{r}}| = |0\mathbf{i} + 3\mathbf{j}| = 3\,\text{m/s}$

And the speed is 3 m/s.

Question 35

$$x + 2y + z = 3 \qquad \text{Eq}^{\text{n}}①$$
$$-x + (p-2)y + (q-1)z = 0 \qquad \text{Eq}^{\text{n}}②$$
$$x + (r+2)y + (s+1)z = 5 \qquad \text{Eq}^{\text{n}}③$$

$\text{Eq}^{\text{n}}①$	$x + 2y + z = 3$
$\text{Eq}^{\text{n}}② + \text{Eq}^{\text{n}}①$	$py + qz = 3$
$\text{Eq}^{\text{n}}③ - \text{Eq}^{\text{n}}①$	$ry + sz = 2$

$y = \dfrac{3 - qz}{p}$

$y = \dfrac{2 - sz}{r}$

$3r - qrz = 2p - psz$

$z(-qr + ps) = 2p - 3r$

$z = \dfrac{2p - 3r}{-qr + ps}$

If $qr = ps$ there is no solution.

In order for there to be a unique solution, $qr \neq ps$.

Question 36

$$\mathbf{r} = \begin{pmatrix} -3 \\ -7 \\ 8 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

Vector from point A to point on line closest to point A:

$$\overrightarrow{AP} = -\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ -7 \\ 8 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 3\lambda_1 - 4 \\ 4\lambda_1 - 7 \\ 12 - 5\lambda_1 \end{pmatrix}$$

The line is parallel to $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ and so $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} 3\lambda_1 - 4 \\ 4\lambda_1 - 7 \\ 12 - 5\lambda_1 \end{pmatrix} = 0$

$9\lambda_1 - 12 + 16\lambda_1 - 28 - 60 + 25\lambda_1 = 0$

$50\lambda_1 = 100 \quad \Rightarrow \quad \lambda_1 = 2$

$$\overrightarrow{AP} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$\left|\overrightarrow{AP}\right| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3\,\text{units}$

The shortest distance from the line $\mathbf{r}$ to the point with position vector $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ is 3 units.

Question 37

a

$a = -g\mathbf{j}\,\text{m/s}^2$

$v = -gt\mathbf{j} + \mathbf{c}$

When $t = 0$, $|v| = u$

$|v| = \sqrt{u^2}$

$\mathbf{c} = -u\sin\theta\,\mathbf{i} + u\cos\theta\,\mathbf{j}$

$v = -u\sin\theta\,\mathbf{i} + (u\cos\theta - gt)\mathbf{j}$

When $t = 0, x = 0\mathbf{i} + 0\mathbf{j}$

$$x = \left[ut\cos\theta\,\mathbf{i} + \left(ut\sin\theta - \frac{gt^2}{2}\right)\mathbf{j}\right]\text{m}$$

b

Given $u = 50\,\text{m/s}$

$g = 10$

$x = 50t\cos\theta\,\mathbf{i} + (50t\sin\theta - 5t^2)\mathbf{j} = 100\mathbf{i} + 40\mathbf{j}$

$50t\cos\theta = 100$

$50t\sin\theta - 5t^2 = 40$

Solving gives $\theta \approx 34.9°,\ 76.9°$.

Question 38

The student's conclusion is not correct. The system of equations has 4 equations with three unknown factors. There is one unique solution. $z = 3,\ y = -1$ and $x = 1$.

Question 39

a $v = -10t\mathbf{j} + \mathbf{c}$

When $t = 0,\ v = 30\mathbf{i} + 24\mathbf{j}$

$\mathbf{c} = 30\mathbf{i} + 24\mathbf{j}$

$v = \left[30\mathbf{i} + (24 - 10t)\mathbf{j}\right]\text{m/s}$

b $x = 30t\mathbf{i} + (24t - 5t^2)\mathbf{j} + \mathbf{c}$

When $t = 0$, its position relative to point T is (0, 0)

$x = \left[30t\mathbf{i} + (24t - 5t^2)\mathbf{j}\right]\text{m}$

c Highest point is reached when $24t - 5t^2$ is at its maximum.

Observe graph or consider when $24 - 10t = 0$.

$t = 2.4\,\text{seconds}$.

d $30t = 135$

$t = 4.5$ seconds

e The greatest height reached by the ball is $24 \times 2.4 - 5 \times 2.4^2 = 28.8\,\text{m}$

f When $t = 4.5$, $x = \left[30 \times 4.5\mathbf{i} + (24 \times 4.5 - 5(4.5)^2)\mathbf{j}\right]\text{m}$

$x = \left[135\mathbf{i} + 6.75\mathbf{j}\right]\text{m}$

$c = 6.75$

Question 40

Using proof by induction.

For $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

When $n = -1$

$$\text{LHS} = (\cos\theta + i\sin\theta)^{-1} = \frac{1}{(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta - i^2\sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$$

$= \cos(-\theta) + i\sin(-\theta)$ [as $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$]

$= \text{RHS}$

Assume that the statement is true for $n = k$, $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

When $n = k-1$, by de Moivre's theorem $(\cos\theta + i\sin\theta)^{k-1} = \cos\left[(k-1)\theta\right] + i\sin\left[(k-1)\theta\right]$

$$\text{LHS} = (\cos\theta + i\sin\theta)^{k-1} = (\cos\theta + i\sin\theta)^k \frac{1}{(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$$

$$= \left[\cos(k\theta) + i\sin(k\theta)\right] \times \left[\cos(-\theta) + i\sin(-\theta)\right]$$

$$= \text{cis}(k\theta) \times \text{cis}(-\theta) = \text{cis}\left[(k-1)\theta\right] = \cos\left[(k-1)\theta\right] + i\sin\left[(k-1)\theta\right]$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta - i^2\sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$$

$= \cos(-\theta) + i\sin(-\theta)$ [as $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$]

$= \text{RHS}$

For a negative integer, it then follows that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, as we have shown that it is true when $n = -1$ and for numbers successively one less than -1 (so negative integers).

Question 41

$$\left(\frac{z_1}{z_2 z_3}\right)^{-3} = \left(\frac{z_2 z_3}{z_1}\right)^3 = \left(\frac{2\,\text{cis}\frac{\pi}{2} \times 3\,\text{cis}\frac{2\pi}{3}}{\sqrt{6}\,\text{cis}\frac{5\pi}{6}}\right)^3 = \left(\frac{6\,\text{cis}\frac{7\pi}{6}}{\sqrt{6}\,\text{cis}\frac{5\pi}{6}}\right)^3 = \left(\sqrt{6}\,\text{cis}\frac{2\pi}{6}\right)^3 = 6\sqrt{6}\,\text{cis}\pi$$

$$= 6\sqrt{6}\cos\pi + i6\sqrt{6}\sin\pi = -6\sqrt{6}$$

Question 42

$$
\begin{aligned}
x+3y-z&=3 \quad \text{Eq}^{\text{n}}① \\
-x-3y+z&=3 \quad \text{Eq}^{\text{n}}② \\
2x+6y-2z&=6 \quad \text{Eq}^{\text{n}}③
\end{aligned}
$$

Eqⁿ①	$x+3y-z=3$
Eqⁿ② + Eqⁿ①	$0=6$
Eqⁿ③ − 2Eqⁿ①	$0=0$

As 0 does not equal 6, there is no solution to this system of linear equations.

Question 43

$$
\begin{aligned}
3x+2y+z&=4 \quad \text{Eq}^{\text{n}}① \\
x-y+2z&=3 \quad \text{Eq}^{\text{n}}② \\
3x+7y+pz&=q \quad \text{Eq}^{\text{n}}③
\end{aligned}
$$

Eqⁿ②	$x-y+2z=3$
Eqⁿ① − 3Eqⁿ②	$5y-5z=-5$
Eqⁿ③ − Eqⁿ①	$5y+(p-1)z=q-4$

Eqⁿ①'	$x-y+2z=3$
Eqⁿ②'	$5y-5z=-5$
Eqⁿ③ − Eqⁿ②'	$(p+4)z=q+1$

a There will be infinite solutions to this system of equations when $p=-4$ and $q=-1$.

b There will be no solutions to this system of equations when $p=-4$ and $q\neq-1$.

c

$$
\begin{aligned}
(k+4)z&=k+8 \\
(k+4)\times 3&=k+8 \\
3k+12&=k+8 \\
2k&=-4 \quad \Rightarrow \quad k=-2
\end{aligned}
$$

$p=-2,\ q=5$

$5y-5(3)=-5 \quad \Rightarrow \quad y=2$

$x-2+2(3)=3 \quad \Rightarrow \quad x=-1$

$m=-1,\ n=2,\ p=-2$ and $q=5$.

Question 44

$$\mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Vector from point *A* to point on line closest to point *A*:

$$\overrightarrow{AP} = -\begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 - 1 \\ \lambda_1 - 8 \\ \lambda_1 - 5 \end{pmatrix}$$

The line is parallel to $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and so $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -\lambda_1 - 1 \\ \lambda_1 - 8 \\ \lambda_1 - 5 \end{pmatrix} = 0$

$$\begin{aligned} \lambda_1 + 1 + \lambda_1 - 8 + \lambda_1 - 5 &= 0 \\ 3\lambda_1 - 12 &= 0 \\ \lambda_1 &= 4 \end{aligned}$$

$$\overrightarrow{AP} = \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix}$$

$$\left|\overrightarrow{AP}\right| = \sqrt{(-5)^2 + (-4)^2 + (-1)^2} = \sqrt{42}\text{ units} \approx 6.48\text{ units}$$

The shortest distance from the path the dog follows to the point where my movement-activated light is situated is 6.48 m. If the dog continues along this path, you would not expect the light to be activated as the dog does not go within 6 m of the light.